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THE DEFORMATION THEORY OF PLASTICITY OF ANISOTROPIC MEDIA*

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Mutually inverse defining equations of the deformation theory of plasticity of media with arbitrary anisotropy are written assuming the relations between the stresses and deformations to be quasilinear. The conditions of plasticity and unloading are considered. Theorems are proved on the existence and uniqueness of solutions of the quasistatic problem of the deformation theory of plasticity and of simple loading. The method of successive approximations for solving the problem is considered, and its convergence is proved. Various means of simplyfying the theory are considered. Theorems of minimum Lagrangian and the maximum of the Castiglianian are proved.

In the deformation theory of plasticity the stresses and deformations are connected by finite relations. When these relations are quasilinear (tensor-linear) /1/, and the medium is isotropic, for simple processes /2/ all theories of plasticity agree with the deformation theory (the theory of small elastic-plastic deformations) /3/. However, in practice that theory is used for a wider class of processes of deformations. The advantage of this theory is its simplicity, the mutually inverse relations between the stresses and deformations, the availability of theorems of existence and uniqueness and of the minimum of the Lagrangian and maximum of the Castiglianian, of the theorem of simple loading and unloading /2/, and also the existence of an effective method of solving quasistatic problems, the method of elastic solutions /2/, whose convergence was adequatly analyzed in /4, 5/. Below a deformation theory is constructed for initially anisotropic media.

1. Let the symmetric stress tensor $\boldsymbol{\sigma}$ be a tensor function of the small deformation tensor $\boldsymbol{\varepsilon}$; this function is invariant to transformations that characterize certain classes of anisotropy. The function can be represented in the form of the dependence of the tensor $\boldsymbol{\varepsilon}$ and some "parametric" tensors $\mathbf{A}_1, \mathbf{A}_2, \ldots$ that define the considered anisotropy class /1/. Let us assume that this anisotropic function is quasilinear (tensor-linear) /1, 6/. This means that its polynomial representation /7/ contains only tensors linearly dependents on $\boldsymbol{\varepsilon}$,

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and tensors independent of it.

Let the form of the quasilinear tensor function in some rectangular Cartesian system of coordinates of three-dimensional Euclidean space have the form

$$\sigma_{ij} = \sum_{\alpha=1}^{n} Y_\alpha (I_1, \ldots, I_n)\, p_{ij}^{(\alpha)} \tag{1.1}$$

where Y_α are some invariant scalar functions of combined invariants $I_1, I_2, \ldots$ of the tensors $\boldsymbol{\varepsilon}, \mathbf{A}_1, \mathbf{A}_2, \ldots,$ and $\mathbf{p}^{(\alpha)}$ is some tensor independent of the deformation tensor ε or dependent on it linearly. In the selected system of coordinates we can write

$$\varepsilon_{ij} = \sum_{\alpha=1}^{n} p_{ij}^{(\alpha)}, \quad \frac{p_{ij}^{(\alpha)} p_{ij}^{(\beta)}}{I_\alpha I_\beta} = \delta_{\alpha\beta}, \quad \frac{\partial I_\alpha}{\partial \varepsilon_{ij}} = \frac{p_{ij}^{(\alpha)}}{I_\alpha} \tag{1.2}$$

i.e. the deformation tensor is expressed in the form of a sum of n pairwise orthogonal tensors $\mathbf{p}^{(\alpha)}$.

It is assumed that summation from 1 to 3 is carried out over the recurrent Latin letter indices, while there is no summation over the Greek letter indices. It can be seen that the number n $(\alpha, \beta = 1, 2, \ldots, n)$ cannot exceed six /1/.

We shall call the invariant $I_\varkappa$ linear, if a constant tensor $\mathbf{a}^{(\varkappa)}$ exists such that

$$p_{ij} = \frac{I_\varkappa}{a_\varkappa} a_{ij}^{(\varkappa)}, \quad a_\varkappa \equiv (a_{ij}^{(\varkappa)} a_{ij}^{(\varkappa)})^{1/2}, \quad \sum_{\varkappa=1}^{m} a_{ij}^{(\varkappa)} = \delta_{ij}, \quad m < n$$

Otherwise the invariant I_γ is called non-linear. We can now represent relations (1.1) in the form

$$\sigma_{ij} = \sum_{\varkappa=1}^{m} Y_\varkappa (I_1, \ldots, I_n) \frac{a_{ij}^{(\varkappa)}}{a_\varkappa} + \sum_{\gamma=m+1}^{n} Y_\gamma (I_1, \ldots, I_n) \frac{p_{ij}^{(\gamma)}}{I_\gamma} \tag{1.3}$$

Without stating this every time, we shall assume that the indices α and β take the values 1 to n, the index $\varkappa$ takes values from 1 to m, and γ takes values from $m+1$ to n.

Relations (1.1) or (1.3) establish a connection between the stresses and defromations when, for instance, the n functions

$$Y_\alpha = Y_\alpha (I_1, \ldots, I_n) = \sum_{\beta=1}^{n} A_{\alpha\beta} [1 - \omega_\alpha (I_1, \ldots, I_n)]\, I_\beta \tag{1.4}$$
$$A_{\alpha\beta} = \frac{C_{ijkl} p_{ij}^{(\alpha)} p_{kl}^{(\beta)}}{I_\alpha I_\beta}$$

where $A_{\alpha\beta}$ is some square matrix $n \times n$ with constant coefficients, C_{ijkl} are the components of the moduli of elasticity tensor, and ω_α are functions of the invariants $I_1, \ldots, I_n$, are known from experiment. For a linearly elastic medium we set $\omega_\alpha = 0$.

If the scalar functions (1.4) are solvable for $I_1, I_2, \ldots, I_n$,

$$I_\alpha = J_\alpha (Y_1, \ldots, Y_n) \equiv \sum_{\beta=1}^{n} B_{\alpha\beta} [1 - \Omega_\alpha (Y_1, \ldots, Y_n)]\, Y_\beta \tag{1.5}$$
$$B_{\alpha\beta} = \frac{J_{ijkl} P_{ij}^{(\alpha)} P_{kl}^{(\beta)}}{Y_\alpha Y_\beta}$$

where J_{ijkl} are the components of the elastic yield tensor, and the tensors $\mathbf{P}^{(\alpha)}$ are defined as in (1.2)

$$\sigma_{ij} = \sum_{\alpha=1}^{n} P_{ij}^{(\alpha)}, \quad \frac{P_{ij}^{(\alpha)} P_{ij}^{(\beta)}}{Y_\alpha Y_\beta} = \delta_{\alpha\beta}, \quad \frac{\partial Y_\alpha}{\partial \sigma_{ij}} = \frac{P_{ij}^{(\alpha)}}{Y_\alpha}$$

then (1.3) are also solvable for the deformations

$$\varepsilon_{ij} = \sum_{\varkappa=1}^{m} I_\varkappa (Y_1, \ldots, Y_n) \frac{a_{ij}^{(\varkappa)}}{a_\varkappa} + \sum_{\gamma=m+1}^{n} I_\gamma (Y_1, \ldots, Y_n) \frac{P_{ij}^{(\gamma)}}{Y_\gamma} \tag{1.6}$$

It is, thus, possible to establish mutually inverse relations between the tensors $\mathbf{P}^{(\alpha)}$ and $\mathbf{p}^{(\alpha)}$

$$P_{ij}^{(\alpha)} = \frac{Y_\alpha}{I_\alpha} p_{ij}^{(\alpha)}, \quad p_{ij}^{(\alpha)} = \frac{I_\alpha}{Y_\alpha} P_{ij}^{(\alpha)}$$

The deviator of the deformation tensor $\mathbf{e}$, that satisfies condition /2/

$$e_{ij}(t) \equiv \varepsilon_{ij}(t) - \frac{1}{3}\theta(t)\delta_{ij} = \lambda(t)e_{ij}^{\circ}, \qquad \theta \equiv \varepsilon_{ii}$$

where $\mathbf{e}^{\circ}$ is a tensor independent of time (of the load parameter), is called the simple deformation process. Similarly for the simple stress process

$$S_{ij}(t) \equiv \sigma_{ij}(t) - \sigma(t)\delta_{ij} = \mu(t)S_{ij}^{\circ}, \quad \sigma \equiv {}^1/_3\sigma_{ii}$$

where $\mathbf{S}^{\circ}$ is the deviator of the stress tensor which is independent of time (of the load parameter).

We call the processes of deformation $\varepsilon(t)$ and of stressing $\sigma(t)$ simple in the narrow sense, if

$$\varepsilon_{ij}(t) = \lambda(t)\varepsilon_{ij}^{\circ}, \quad \sigma_{ij}(t) = \mu(t)\sigma_{ij}^{\circ}$$

We call the process of defromation and stressing, respectively, simple in the broad sense, if

$$p_{ij}^{(\gamma)}(t) = \lambda(t)\overset{\circ}{p}{}_{ij}^{(\gamma)}, \qquad P_{ij}^{(\gamma)}(t) = \mu(t)\overset{\circ}{P}{}_{ij}^{(\gamma)}$$

It will be seen that to the simple process of defromation in the conventional, broad and narrow sense there corresponds the stressing process in the conventional, broad and narrow senses, respectively.

In the deformation space let the function $\varphi(I_1, \ldots, I_n)$ be specified, and the quantity φ_0, which is experimentally determined and may depend on the loading history /3, 8/. The condition of plasticity involves the following. If

$$\varphi(I_1, \ldots, I_n) < \varphi_0 \tag{1.7}$$

the relation between the stresses and strains (1.3), (1.6) obeys Hooke's law, i.e. it is necessary to set $\omega_\alpha \equiv 0, \Omega_\alpha \equiv 0$ in (1.4) and (1.5). If, however, inequality (1.7) is violated, plastic deformation takes place. If the process is active (loading) ($\sigma_{ij}d\varepsilon_{ij} = Y_1 dI_1 + \ldots$ $Y_n dI_n$ or, for example, $d\varphi > 0$), (1.4) hold. If, however, at some instant unloading begins (passive process), ($\sigma_{ij}d\varepsilon_{ij} = Y_1 dI_1 + \ldots + Y_n dI_n < 0$ or, for example, $d\varphi < 0$), then it is necessary instead of (1.4) to use the conditions of, for instance, linear unloading

$$Y_\alpha - Y_\alpha' = \sum_{\beta=1}^{n} A_{\alpha\beta}(I_\beta - I_\beta') \tag{1.8}$$

where the quantities denoted by a prime correspond to stresses and deformations accumulated up to the instant when unloading begins.

Obviously the condition of plasticity (1.7) may also be formulated in the stress space

$$\Phi(Y_1, \ldots, Y_n) < \Phi_0 \tag{1.9}$$

Then at unloading $(\varepsilon_{ij}d\sigma_{ij} \equiv I_1 dY_1 + \ldots + I_n dY_n < 0$ or $d\Phi < 0)$ instead of (1.8) we have

$$I_\alpha - I_\alpha' = \sum_{\beta=1}^{n} B_{\alpha\beta}(Y_\beta - Y_\beta') \tag{1.10}$$

2. The equations of equilibrium

$$\sigma_{ij,j} + X_i = 0 \tag{2.1}$$

where $\mathbf{X}$ is the vector of volume forces, can be written, using (1.4) in the form

$$\sum_{\varkappa=1}^{m} Y_{\varkappa,j}\frac{a_{ij}^{(\varkappa)}}{a_\varkappa} + \Big[\sum_{\gamma=m+1}^{n} Y_\gamma \frac{p_{ij}^{(\gamma)}}{I_\gamma}\Big]_{,j} + X_i = 0 \tag{2.2}$$

Suppose that on the part Σ_1 of the body boundary Σ that bounds the volume V, we are given, for instance, the displacements $u_i^{\circ} = 0$, and on another part Σ_2 the load S_i°

$$u_i|_{\Sigma_1} = 0, \qquad \sigma_{ij}n_j|_{\Sigma_2} = S_i^{\circ} \tag{2.3}$$

Then the quasistatic problem of the deformation theory of plasticity (in displacements) consists of solving the equilibrium equations (2.1) and satisfying the boundary conditions (2.3), taking condition (1.4) or (1.8) into account (depending on whether loading or unloading takes place) and the Cauchy relations

$$\varepsilon_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i}) \tag{2.4}$$

We shall call problem (2.2), (2.3), (1.4) or (1.8), (2.4) problem A.

The quasistatic problem in terms of stresses (problem B) consists of solving the equations of equilibrium (2.1) and six equations of compatibility

$$\eta_{ij} \equiv \epsilon_{ikl}\,\epsilon_{jmn}\varepsilon_{kn,\,lm} = 0 \tag{2.5}$$

when the boundary conditions (2.3) and satisfied. In (2.5) and (2.3) the deformations (in displacements) must be expressed in stresses in agreement with (1.6) and (1.5) or (1.10). The quasistatic problem C can also be formulated in the deformation theory of plasticity in stresses /9/.

A medium is said to be anisotropically incompressible, when the conditions

$$I_\varkappa \equiv 0, \quad \varkappa = 1, \ \ldots, \ m \tag{2.6}$$

hold. For an anistoropically incompressible medium in problem A, m equations (2.6) are added for the m unknowns $Y_\varkappa, \varkappa = 1, \ldots, m$ in the equilibrium equations (2.2), since in (1.4) only relations

$$Y_\gamma = Y_\gamma(I_{m+1}, \ldots, I_n) \equiv \sum_{\delta=m+1}^{n} A_{\gamma\delta}\,[1 - \omega_\gamma(I_{m+1}, \ldots, I_n)]\,I_\delta$$

remain.

To solve the boundary value problems, for instance problem A, the method of successive approximations may be used

$$C_{ijkl}u_{k,\,lj}^{(q+1)} = C_{ijkl}u_{k,\,lj}^{(q)} - \beta\,[\sigma_{ij,\,j}(\mathbf{u}^{(q)}) + X_i] \tag{2.7}$$

$$u_i^{(q+1)}|_{\Sigma_1} = 0, \qquad C_{ijkl}u_{k,\,l}^{(q+1)}n_j|_{\Sigma_2} = C_{ijkl}u_{k,\,l}^{(q)}n_j|_{\Sigma_2} - \beta\,[\sigma_{ij}(\mathbf{u}^{(q)})\,n_j|_{\Sigma_2} - S_i^\circ] \tag{2.8}$$

where the expression $\sigma_{ij}(\mathbf{u})$ denotes that the stresses are expressed in terms of deformations by relations (1.3) and (1.4) or (1.8), and the deformations in terms of relations (2.4).

We will say that a material possesses a soft characteristic with respect to the invariant I_α, if

$$\frac{Y_\alpha}{I_\alpha} \geqslant \frac{\partial Y_\alpha}{\partial I_\alpha}, \qquad \frac{I_\alpha}{Y_\alpha} \leqslant \frac{\partial I_\alpha}{\partial Y_\alpha} \tag{2.9}$$

and a stiff characteristic with respect to that invariant, if

$$\frac{\partial Y_\alpha}{\partial I_\alpha} \geqslant \frac{Y_\alpha}{I_\alpha}, \qquad \frac{\partial I_\alpha}{\partial Y_\alpha} \leqslant \frac{I_\alpha}{Y_\alpha} \tag{2.10}$$

(Note that the conditions of linear unloading (1.8) are only valid for materials with a soft characteristic.)

We put

$$D_\alpha \equiv I_\alpha \sum_{\substack{\beta=1 \\ (\alpha\neq\beta)}} \frac{1}{I_\beta}\left|\frac{\partial Y_\beta}{\partial I_\alpha}\right|, \qquad G_\alpha \equiv Y_\alpha \sum_{\substack{\beta=1 \\ (\alpha\neq\beta)}} \frac{1}{Y_\beta}\left|\frac{\partial I_\beta}{\partial Y_\alpha}\right|$$

and suppose that in the case of a soft characteristic with respect to the invariant I_α the following inequalities are satisfied:

$$0 < m_0 \leqslant \frac{\partial Y_\alpha}{\partial I_\alpha} - D_\alpha \leqslant \frac{Y_\alpha}{I_\alpha} + D_\alpha \leqslant M_0 \tag{2.11}$$

$$0 < n_0 \leqslant \frac{I_\alpha}{Y_\alpha} - G_\alpha \leqslant \frac{\partial I_\alpha}{\partial Y_\alpha} + G_\alpha \leqslant N_0 \tag{2.12}$$

while for the case of a stiff characteristic with respect to the same invariant the inequalities

$$0 < m_0 \leqslant \frac{Y_\alpha}{I_\alpha} - D_\alpha \leqslant \frac{\partial Y_\alpha}{\partial I_\alpha} + D_\alpha \leqslant M_0 \tag{2.13}$$

$$0 < n_0 \leqslant \frac{\partial I_\alpha}{\partial Y_\alpha} - G_\alpha \leqslant \frac{I_\alpha}{Y_\alpha} + G_\alpha \leqslant N_0 \tag{2.14}$$

are satisfied where m_0, M_0, n_0, N_0 are certain positive numbers.

Theorem 2.1. Suppose a unique generalized solution exists of the problem of the theory of elasticity of an anisotropic body $\mathbf{u}^{(0)}$ that is obtained from problem A when $\omega_\alpha \equiv 0$ ($\alpha = 1, \ldots, n$). Moreover suppose inequalities (2.11) or (2.13) hold, and the volume and surface forces belong to the space L_q /10/, and

$$X_i \in L_q(V),\ q > {}^6/_5;\ S_i^\circ \in L_q(\Sigma_2),\ q > {}^4/_3$$

Then a unique generalized solution $\mathbf{u}^*$ of problem A exists, and for any value of the iteration parameter β $(0 < \beta < {}^2/M_0)$ and the process of successive approximations (2.7), (2.8) beginning with $\mathbf{u}^{(0)}$ converges to it, and

$$\| \mathbf{u}^{(q)} - \mathbf{u}^* \|_{L_2} \leqslant q_0^q \| \mathbf{u}^{(0)} - \mathbf{u}^* \|_{L_2}$$
$$q_0 \equiv \max(|1 - \beta m_0|,\ |1 - \beta M_0|)$$

Then the quantity q_0 attains its minimum value $(M_0 - m_0)/(M_0 + m_0)$ when $\beta = 2/(M_0 + m_0)$.
The proof of this theorem follows as a special case of the theorem proved in /5, 9/. In exactly the same way we can formulate the theorem of the existence and uniqueness of the solution of problem B and of the convergence of the method of successive approximations using inequalities (2.12) or (2.14).
The theorems on simple loading also hold.

Theorem 2.2. Let the specified volume and surface forces incrase in proportion to one parameter $\mu(t)$

$$X_i(t, \mathbf{x}) = \mu(t) X_i^\circ(\mathbf{x}),\ S_i^\circ(t, \mathbf{x}) = \mu(t) S_i^\infty(\mathbf{x}) \tag{2.15}$$

while on the boundary Σ the displacements u_i° are proportional to another parameter $\lambda(t)$

$$u_i^\circ(t, \mathbf{x}) = \lambda(t) u_i^\infty(\mathbf{x}) \tag{2.16}$$

Suppose further that the functions (1.4) are power functions

$$Y_\alpha(I_1, \ldots, I_n) = \sum_j c_{\alpha j} I_1^{k_{\alpha 1 j}} \ldots I_n^{k_{\alpha n j}} \tag{2.17}$$

all $k_{\alpha ij}$ $(i = 1, \ldots, n)$ are non-negative numbers $c_{\alpha j} \neq 0$, and summation is carried out over j such that

$$\sum_{i=1}^n k_{1ij} = \ldots = \sum_{i=1}^n k_{nij} = r$$

where r is a fixed positive number.
The processes of stressing and deformation are then simple (in any sense) at every point of the medium, if

$$\mu(t) = [\lambda(t)]^r \tag{2.18}$$

The theorem is proved by the method proposed in /2/.

Theorem 2.3. Let the medium be anisotropic, incompressible (2.6) and let conditions (2.15) and (2.16) be satisfied. The stressing and deformation processes are then simple (in the wide sense) at each point of the medium if

$$Y_\gamma(I_{m+1}, \ldots, I_n) = \sum_j c_{\gamma j} I_{m+1}^{k_{\gamma m+1 j}} \ldots I_n^{k_{\gamma n j}} \tag{2.19}$$

Here $k_{\gamma ij}$ $(i = m+1, \ldots, n)$ are non-negative numbers $c_{\gamma j} \neq 0$, and summation is carried out over j such, that

$$\sum_{i=m+1}^n k_{m+1ij} = \ldots = \sum_{i=m+1}^n k_{nij} = r$$

where r is a fixed positive number, and condition (2.18) is satisfied.

3. Thus to use the above theory it is necessary to know the n $(n \leqslant 6)$ experimentally obtained functions (1.4) or (1.5) of n variables. The problem of the experimental determination of these functions is fairly complicated. In this connection it is possible to consider a simplified version of the theory according to which all linear invariants are related by Hook's law. Instead of (1.4) we then have

$$Y_\varkappa = \sum_{\delta=1}^m A_{\varkappa\delta} I_\delta \tag{3.1}$$
$$Y_\gamma = Y_\gamma(I_{m+1}, \ldots, I_n) \equiv \sum_{\delta=m+1}^n A_{\gamma\delta}[1 - \omega_\gamma(I_{m+1}, \ldots, I_n)] I_\delta$$

Similarly, instead of (1.5) we have

$$I_\varkappa = \sum_{\delta=1}^{m} B_{\varkappa\delta} Y_\delta$$

$$I_\gamma = I_\gamma(Y_{m+1}, \ldots, Y_n) \equiv \sum_{\delta=m+1}^{n} B_{\gamma\delta}[1 - \Omega_\gamma(Y_{m+1}, \ldots, Y_n)]\, Y_\delta$$

where the matrices with elements $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are mutually inverse.

The conditions of plasticity (1.7) and (1.9) for the simplified theory can be written respectively, in the form

$$\varphi(I_{m+1}, \ldots, I_n) < \varphi_0, \quad \Phi(Y_{m+1}, \ldots, Y_n) < \Phi_0$$

i.e. the first m invariants do not affect the plastic region. When the conditions

$$Y_\gamma = Y_\gamma(I_\gamma) \equiv \sum_{\delta=m+1}^{n} A_{\gamma\delta}[1 - \omega(I_\gamma)]\, I_\delta$$

$$I_\gamma = I_\gamma(Y_\gamma) \equiv \sum_{\delta=m+1}^{n} B_{\gamma\delta}[1 - \Omega(Y_\gamma)]\, Y_\delta$$

are satisfied, the simplified theory may be called the simplest.

To solve the related problems of thermoplasticity it is necessary to obtain the dissipation functions W^* /11/. For the theory considered here this function has the form

$$W^* = \sum_{\alpha=1}^{n} Y_\alpha \frac{d}{dt}[\omega_\alpha I_\alpha] \tag{3.2}$$

and for the simplified theory (3.1) it is

$$W^* = \sum_{\alpha=m+1}^{n} Y_\gamma \frac{d}{dt}[\omega_\gamma I_\gamma] \tag{3.3}$$

Formulas (3.2) and (3.3) may also be used to construct a theory of strength.

4. It is sometimes assumed that the deformation theory of plasticity is identical with the physically non-linear theory of elasticity /12/ in the case of the active process. That assumption is equivalent to the potentiality of tensor (1.1) /2, 9/, i.e. a scalar function $W(I_1, \ldots, I_n)$ exists such that

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = \sum_{\alpha=1}^{n} \frac{\partial W}{\partial I_\alpha} \frac{\partial I_\alpha}{\partial \varepsilon_{ij}} \tag{4.1}$$

From a comparison of (4.1) and (1.1), (1.2) it follows that

$$Y_\alpha = \partial W / \partial I_\alpha \tag{4.2}$$

and hence the following $n(n-1)/2$ relations must exist between functions (1.4):

$$\partial Y_\alpha / \partial I_\beta = \partial Y_\beta / \partial I_\alpha, \quad \alpha < \beta \tag{4.3}$$

Since the matrix $A_{\alpha\beta}$ is symmetric, the number of such functions for the simplified theory is reduced to $(n-m)(n-m-1)/2$, and for the simplest theory all formulas of type (4.3) are satisfied identically.

If the tensor (1.1) is potential, the deformation tensor (1.6) must be such also, i.e. a scalar function $w(Y_1, \ldots Y_n)$ exists such that

$$\varepsilon_{ij} = \frac{\partial w}{\partial \sigma_{ij}} = \sum_{\alpha=1}^{n} \frac{\partial w}{\partial Y_\alpha} \frac{\partial Y_\alpha}{\partial \sigma_{ij}}, \quad I_\alpha = \frac{\partial w}{\partial Y_\alpha} \tag{4.4}$$

If $W(0) = 0, w(0) = 0$, then the following identity /9/ also holds

$$W + w = \sigma_{ij} e_{ij}$$

For potential stress and deformation tensors we can construct the Lagrangian and the Castiglianian

$$L \equiv \int_V W(\mathbf{u})\, dV - \int_V X_i u_i\, dV - \int_{\Sigma_s} S_i^\circ u_i\, d\Sigma \tag{4.5}$$

$$K \equiv -\int_V w(\boldsymbol{\sigma})\, dV + \int_{\Sigma_1} \sigma_{ij} n_j u_i^\circ\, d\Sigma \tag{4.6}$$

Using them, we can formulate the Lagrange and Castigliano variational principle /9/.

Theorem 4.1. When inequalities (2.9), (2.11) or (2.10), (2.13) in which (4.1) is taken into account are satisfied, the Lagrangian (4.6) has its minimum in the equilibrium position. The equilibrium point is unique.

Theorem 4.2. When inequalities (2.9), (2.12) or (2.10), (2.14) in (4.4) is taken into account are satisfied, the Castiglianian (4.6) at the equilibrium position has a maximum. The maximum point is unique.

Theorem 4.3. In the position of equilibrium the Castiglianian is identical with the Lagrangian.

The proof of Theorems 4.1—4.3 follows from /9/, as a special case.

For potential stress and strain tensors Theorems 2.2 and 2.3 on simple loading remain valid, except that instead of constraints (2.17) and (2.19) the following must be taken, respectively:

$$W=\sum_q c_q I_1^{k_{1q}}\ldots I_n^{k_{nq}}\left(W=\sum_q c_q I_{m+1}^{k_{m+1q}}\ldots I_n^{k_{nq}}\right),\quad \sum_q c_q^2\neq 0$$

where k_{iq} are non-negative numbers, and summation is carried out over q such that

$$\sum_{i=1}^{n} k_{iq}=r+1\quad\left(\text{or}\ \sum_{i=m+1}^{n} k_{iq}=r+1\right)$$

where r is a fixed non-negative number.

The theory is also simplified by the possible assumption that the function $\varphi(I_1,\ldots,I_n)$ in (1.7) has the form

$$\varphi=W=\int\sum_{\alpha=1}^{n} Y_\alpha\, dI_\alpha$$

or in the case of the simplified theory

$$\varphi=W'=\int\sum_{\gamma=m+1}^{n} Y_\gamma\, dI_\gamma$$

5. Consider the special case of a transversely isotropic medium. In this case $n=4$, $m=2$ and we can assume that the transverse isotropy axis is directed along x_3. We then have

$$a_{ij}^{(1)}=\delta_{i1}\delta_{j1}+\delta_{i2}\delta_{j2},\quad a_1=\sqrt{2},\quad a_{ij}^{(2)}=\delta_{i3}\delta_{j3},\quad a_2=1$$

$$I_1=\frac{\sqrt{2}}{2}(\varepsilon_{11}+\varepsilon_{22}),\quad I_2=\varepsilon_{33}$$

$$I_3=\frac{\sqrt{2}}{2}\sqrt{(\varepsilon_{11}-\varepsilon_{22})^2+4\varepsilon_{12}^2},\quad I_4=\sqrt{2(\varepsilon_{13}^2+\varepsilon_{23}^2)}$$

$$p_{ij}^{(1)}=\frac{1}{2}(\varepsilon_{11}+\varepsilon_{22})(\delta_{i1}\delta_{j1}+\delta_{i2}\delta_{j2})$$

$$p_{ij}^{(2)}=\varepsilon_{33}\delta_{i3}\delta_{j3},\quad p_{ij}^{(3)}=\varepsilon_{ij}-\frac{1}{2}(\varepsilon_{11}+\varepsilon_{22})(\delta_{i1}\delta_{j1}+\delta_{i2}\delta_{j2})+\varepsilon_{33}\delta_{i3}\delta_{j3}-$$
$$\varepsilon_{i3}\delta_{j3}-\varepsilon_{j3}\delta_{i3},\quad p_{ij}^{(4)}=\varepsilon_{i3}\delta_{j3}+\varepsilon_{j3}\delta_{i3}-2\varepsilon_{33}\delta_{i3}\delta_{j3}$$

The non-zero independent components of the symmetric matrix $A_{\alpha\beta}$ are

$$A_{11}=C_{1111}+C_{1122},\ A_{22}=C_{3333}$$
$$A_{33}=C_{1111}-C_{1122}\equiv 2C_{1212}$$
$$A_{44}=2C_{1313},\quad A_{12}=\sqrt{2}\,C_{1313}$$

In the case of simplified theory the relations between the stress and strain invariants have the form

$$Y_\xi=A_{\xi1}I_1+A_{\xi2}I_2,\quad \xi=1,\ 2 \tag{5.1}$$
$$Y_\eta=Y_\eta(I_3,\ I_4)\equiv A_{\eta\eta}\,[1-\omega(I_3,\ I_4)]\,I_\eta,\quad \eta=3,\ 4$$

When the stress and strain tensors are potential, the following relation exists between ω_3 and ω_4:

$$A_{33}\left[1-\omega_3-I_3\frac{\partial\omega_3}{\partial I_4}\right]=A_{44}\left[1-\omega_4-I_4\frac{\partial\omega_4}{\partial I_3}\right] \tag{5.2}$$

The conditions of anistropic incompressibility (2.6) for a transversely isotropic medium $I_1=I_2=0$ mean that the volume of the medium does not change under deformation, and that there are no defromations in the direction of the x_3 axis. For a laminar medium with isotropic layers, that may be in some cases modelled as a homogeneous transversely isotropic medium /9/, the axis is perpendicular to the layers. The first two terms in (5.1) indicate that plastic deformations do not occur when there is a change in volume and when the deformation is in the direction of the x_3 axis. For the simplest theory, (5.2) is satisfied identically, and inequalities (2.11) for the soft characteristic with respect to invariants I_3 and I_4 have the form

$$0<\omega_\alpha \leqslant \omega_\alpha + I_\alpha \frac{\partial \omega_\alpha}{\partial I_\alpha} \leqslant \eta_\alpha < 1, \qquad \alpha = 3, 4$$

6. Consider an isotropic medium. In this case $n = 2$, $m = 1$ and

$$a_{ij}^{(1)} = \delta_{ij}, \qquad a_1 = \sqrt{3}, \qquad I_1 = {}^1/_3\sqrt{3}\,\theta$$
$$I_2 = \varepsilon_u \equiv \sqrt{e_{ij}e_{ij}}, \qquad p_{ij}^{(1)} = \frac{1}{3}\theta\delta_{ij}, \qquad p_{ij}^{(2)} = e_{ij}$$

Matrix $A_{\alpha\beta}$ is diagonal, and (λ and μ are Lamé constants)

$$A_{11} = 3\lambda + 2\mu, \quad A_{22} = 2\mu$$
$$\mu \equiv C_{1212}, \quad \lambda \equiv C_{1111} - 2C_{1212} \equiv C_{1122}$$

Inequalities (2.11) for a soft characteristic with respect to invariants I_1 and I_2 have the form /5/

$$0<m_0 \leqslant 3\frac{\partial\sigma}{\partial\theta} - \frac{\theta}{\varepsilon_u}\left|\frac{\partial\sigma_u}{\partial\theta}\right| \leqslant 3\frac{\sigma}{\theta} + \frac{\theta}{\varepsilon_u}\left|\frac{\partial\sigma_u}{\partial\theta}\right| \leqslant M_0$$
$$0<m_0 \leqslant \frac{\partial\sigma_u}{\partial\varepsilon_u} - 3\frac{\varepsilon_u}{\theta}\left|\frac{\partial\sigma}{\partial\varepsilon_u}\right| \leqslant \frac{\sigma_u}{\varepsilon_u} + 3\frac{\varepsilon_u}{\theta}\left|\frac{\partial\sigma}{\partial\varepsilon_u}\right| \leqslant M_0$$

The simplified theory in this case is identical with the simplest theory and with the Il'yushin theory of small elastic-plastic deformations /2/ ($\omega_2 \equiv \omega$)

$$\sigma = (\lambda + {}^2/_3\mu)\,\theta, \quad s_{ij} = 2\mu\,[1 - \omega(\varepsilon_u)]\,e_{ij}$$

Inequality (2.11) for a soft characteristic of the material is identical with the Il'yushin inequality

$$0<\omega \leqslant \omega + \varepsilon_u\frac{d\omega}{d\varepsilon_u} \leqslant \eta < 1, \qquad \eta \equiv 1 - \frac{m_0}{2\mu}, \qquad M_0 \equiv 2\mu$$

Note that in the simplified theory the stress tensor (and hence also the deformation tensor) is always potential

$$W = \frac{1}{2}\left(\lambda + \frac{2}{3}\mu\right) + \int_0^{\varepsilon_u} \sigma_u d\varepsilon_u$$
$$w = \frac{1}{2}\left(\lambda + \frac{2}{3}\mu\right)^{-1} + \int_0^{\sigma_u} \varepsilon_u d\sigma_u$$

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